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A third order convergent method for finding roots of nonlinear equations

Given a function $f(x)$ with a nonzero derivative in the neighborhood of the root, the Newton's method is derived by approximating the area described by the indefinite integral of the derivative of the function in

$$f(x) = f(x_n) + \int_{x_n}^{x} f^{(1)}(\theta)d\theta$$

by a rectangle. This gives rise to the local linear model:

$$M_n(x) = f(x_n) + f^{(1)}(x_n)(x - x_n)$$

In this paper, we suggest a method to approximate this area by the more sophisticated Gaussian Quadrature formula with two nodes. Consequently, we get the nonlinear model:

$$f(x) = f(x_n) + \frac{(x - x_n)}{2}\left[f^{(1)}\{ax_n + bx\} + f^{(1)}\{bx_n + ax\}\right],$$

where $a = \frac{1}{2} + \frac{1}{2\sqrt{3}}$ and $b = \frac{1}{2} - \frac{1}{2\sqrt{3}}$

The resulting iterative scheme is

$$x_{n+1} = x_n - \frac{2f(x_n)}{f^{(1)}\{ax_n + bx^*_{n+1}\} + f^{(1)}\{bx_n + ax^*_{n+1}\}}$$, which is implicit.

Our new scheme is coupled with Newton's method given by

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f^{(1)}(x_n)}$$

and hence we call it Gaussian Quadrature Newton's Method (GQNM). We can show that GQNM is third order convergent. We tested the method for functions such as trigonometric, exponential and combinations of those two as well. The computed results support the established theory.