E1-34: Uniqueness of classifying spaces: a methodical attack

Kirthi Premadasa
(*University of Colombo, Colombo 3*)

Two of the key problems in the subject of Algebraic Topology are the existence problem and the uniqueness problem. Looking at the problem from a Cohomological point of view, the problem can be stated in the following way. Given a Ring R whether there is a topological space X such that the Cohomology ring of X is R and if so, how uniquely does R determine X? Classifying Spaces of Compact Lie Groups is one class of spaces, for which, this problem has special significance (These have "nice" Cohomology rings). Let G be a compact Lie Group and p be a prime. The uniqueness problem can be stated as:
Let X be a

Z/pZ complete space such that $H^*(X, Z/pZ) \cong H^*(BG\hat{p}, Z/pZ)$.

Then is X Homotopy equivalent to $BG\hat{p}$? The problem was successfully answered in the affirmative for the case when $p \mid o(W)$ where W is the weyl group of G.

For the cases where $p \mid o(W)$, though there have been several proofs for specific Lie Groups, no general method of attack is known. We propose a general method of attack for the problem. We first write both the unknown space X and the space $BG\hat{p}$ as homotopy limits of smaller pieces (In the case of $BG\hat{p}$, the pieces are the p completed classifying spaces of the centralizers of the elementary p subgroups of G). Then we prove that the pieces of the two diagrams are homotopy equivalent to each other. Then we propose a way in which, these homotopy equivalences can be collected together to form a map up to homotopy between the two diagrams. Then this map is "rigidified" to get a map in the Topological Category between the two diagrams which, in turn, will yield the required homotopy equivalence between X and $BG\hat{p}$. We use this method to give a short proof for the uniqueness of $BSO(3)\hat{}_2$. We believe that the proof can be extended to attack the unsolved $BSO(2n+1)\hat{}_2$ case.