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# Laguerre Series Expansion for the Probability Density Function of Sum of Independent Random Variables Defined Over $[0, \infty)$

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## ABSTRACT

*The evaluation of the probability density function of sum of independent and identically distributed random variables has been of both theoretical and practical interest for many decades. This paper addresses the evaluation of the density function of sum of random variables defined over the positive real axis. A series expansion for approximating the density function is derived using modified Laguerre polynomials. The newly derived series expansion is exact for Rayleigh distributions. It can also be used to determine the PDF of the sum when the individual random variables are non-identically distributed. The cumulative distribution function is obtained from the series expansion for the probability density. The convergence of the series is established. The numerical results are presented for the density functions and cumulative distribution functions for the sum of Nakagami random variables and Rayleigh random variables.*

**Keywords:** Approximation methods, Laguerre series, Mathematical statistics, Sum of random variables, Probability density functions

## INTRODUCTION

The evaluation of the probability density function (PDF) of sum of independent and identically distributed (iid) random variables arises often in many practical problems in engineering and science (Proakis, 1995). Although several methods have been documented in the literature to solve specific problems that involve the distribution of sum of random variables, there are many problems for which these ad hoc solutions are not applicable. The evaluation of the PDF of sum of random variables in a more general form that can be applied to wide variety of problems that occur in many research areas therefore remains a challenging problem.

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The research relating to the approximation of nearly Gaussian random variables using an infinite series goes far back as the middle of the nineteenth century (Blinnikov and Moessner, 1998). Over the last two centuries, three main expansions using the Hermite polynomials have been developed, namely, the Gram-Charlier, Gauss-Hermite, and Edgeworth expansions. An equally important problem is the expansion of the PDF of the sum of non-Gaussian random variables. The past attempts to extend the expansions developed for non-Gaussian PDFs to approximate the PDF of the sum of random variables have not been very successful. More recently, Beaulieu (Beaulieu, 1990) has derived an infinite series in terms of sinusoidal functions for expanding the cumulative distribution function (CDF) of sum of independent random variables but it has not gained much popularity.

In this paper, we specifically consider the general problem of determining the PDF $f_X(x)$ of a continuous random variable $X$ that results from summing $n$ independent random variables $X_1, X_2, \ldots\ldots, X_n$ whose domain is $[0,\infty)$, that is,

$$X = \sum_{i=1}^{n} X_i \tag{1}$$

Our interests are mainly in the sum of Rayleigh, Nakagami, Weibull and gamma type random variables. It is well known that when two independent random variables are summed together, the PDF of the resulting random variable is obtained by convolving the PDFs of the two random variables. The convolution is practically possible only when two or three random variables having PDFs in very simple mathematical form are involved. If the PDF is in a more complex form like in the case of Nakagami and Weibull distributions or when the number of random variables involved is large, the convolution becomes mathematically intractable. For such cases, an alternative approach is to express the PDF of the sum in a suitable series form. According to the Central Limit theorem (Cramer, 1954), the PDF of the sum of iid random variables approaches the normal distribution as the number of random variables becomes large. This has led to the development of the Gram-Charlier (Cramer, 1954) type series expansions which are belonging to the family of orthogonal series expansions based on Hermite polynomials. Although such expansions have been widely used to model non-Gaussian distributions satisfactorily, they are characterized by several undesirable properties such as slow convergence and poor accuracy (Blinnikov and Moessner, 1998). Therefore, they are not particularly suitable for problems that involve very low probabilities, such as those encountered in digital communications where the probabilities are typically of the order of $10^{-4}$ or lower. The aim of the research reported in this paper is to develop a series expansion as an alternative to the Gram-Charlier series using the Laguerre polynomials to approximate the PDF of the sum of random variables that are approximately in a similar form to a Rayleigh distribution. Previously the Laguerre series expansion has been employed to approximate the PDF of single non-Gaussian random variables. The resulting series is applied to obtain the PDF of the sum of independent random variables. The individual random variables need not have identical distributions.

The organization of the remainder of the paper is as follows. In Section 2, the Laguerre series is introduced and the modified Laguerre series for expanding the PDF of sum of random variables is developed in Section 3. A procedure for computing the coefficients of the new series is developed in Section 4 and an expression for the CDF is given in Section 5. The convergence of the series is treated in Section 6 and an application of the series to determine the CDF and PDF of Nakagami sum is considered in Section 7. Finally, Section 8 with conclusions terminates the paper.

## LAGUERRE SERIES

It is well known that a continuous function $f_X(x)$ can be approximated by an infinite series using a set of orthogonal basis functions $\{\phi_k(x), k = 0,1......\infty\}$ as

$$f_X(x) = \sum_{k=0}^{\infty} C_k \phi_k(x) \tag{2}$$

where $C_k$ are the coefficients of the series. If Hermite polynomials are used as an orthonormal basis to expand $f_X(x)$ for $-\infty < x < \infty$, the resulting expansion is called the Hermite series. The Laguerre polynomials (Singroura, 1963; Lebedev, 1965; Cohen, 1992), $\{L_k(x), k = 0,1,......\infty\}$ form an orthogonal basis in the domain $[0,\infty)$ and therefore they too can be used for the series expansion of a continuous function for $0 \le x < \infty$. The resulting series is called the Laguerre series, which has the form

$$f_X(x) = \sum_{k=0}^{\infty} C_k L_k(x) \tag{3}$$

The Laguerre polynomials are given in the compact form (Abramowitz and Stegum, 1972).

$$L_n(x) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{x^l}{l!} \tag{4}$$

and they are orthonormal with respect to the weighting function $e^{-x}$, that is

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\, dx = \delta_{nm} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \tag{5}$$

The coefficients $C_k$ in (3) can therefore be computed as

$$C_k = \int_0^{\infty} f_X(x)\; L_k(x)\; dx \tag{6}$$

## MODIFIED LAGUERRE SERIES

Our main objective is to expand the PDFs that are in similar form to a Rayleigh density whose domain is $[0, \infty)$. It is therefore desirable if the weighting function can be selected to be a valid PDF such as Rayleigh in an analogues manner to the Gram-Charlier series in which the weighting function is the normal distribution $e^{-x^2/2}/\sqrt{2\pi}$. To this end, if the integral in (5) is transformed using the transformation $x = t^2$, we get

$$\int_{0}^{\infty} 2t\,e^{-t^2} L_m\left(t^2\right) L_n\left(t^2\right) dt = \delta_{mn} \tag{7}$$

Equation (7) implies that the polynomials $L_n\left(t^2\right)$ are also orthonormal but with respect to the weighting function $2\,t\,e^{-t^2}$. We recall that this weighting function is exactly the Rayleigh PDF having a unity second moment. Thus, it is convenient to use the orthonormal basis $\{L_k(x^2),\, k = 0,1,2\ldots\ldots\infty\}$ to expand the PDF $f_X(x)$ as

$$f_X(x) = \sum_{k=0}^{\infty} 2\,x\,e^{-x^2}\, C_k\, L_k\left(x^2\right) \tag{8}$$

This form of the Laguerre series will be referred to as the modified Laguerre series in what follows. A similar series has been obtained (Abdi *et al.*, 2000) in evaluating an integral expression for the PDF of random vectors using the generating function for the Laguarre polynomials and the Hankel transform following lengthy process. The coefficients $C_k$ can be expressed using the orthogonal property of the modified Laguerre polynomials as

$$C_k = \int_{0}^{\infty} f_X(x) L_k\left(x^2\right) dx \tag{9}$$

We observe that the modified Laguerre series in (8) is exact for the Rayleigh distribution in the sense that the series needs only a single term to expand a Rayleigh PDF. Since $f_X(x)$ is not known, it is necessary to develop a procedure to evaluate $C_k$ without the knowledge of $f_X(x)$ which is considered in the next section in detail.

## EVALUATION OF COEFFICIENTS $C_k$

The coefficients $C_k$ can be expressed in terms of the moments of $X$ by substituting in (9) for $L_k(x^2)$ from (4) and then interchanging the integration and summation as

$$C_k = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \frac{1}{l!} \int_0^{\infty} x^{2l} f_X(x)\, dx \tag{10}$$

The integral in (10) yields the $(2l)^{th}$ moment of the random variable $X$. Denoting this as $M_{2l}$, the coefficients $C_k$ can be expressed as

$$C_k = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \frac{1}{l!} M_{2l} \tag{11}$$

### *Evaluation of moments* $M_{2l}$

As mentioned in the previous section, since the PDF of $X$ is not known, an indirect approach is necessary to obtain the moments $M_{2l}$. In Gram-Charlier type series, the moments are normally evaluated from the characteristic function (CHF) of $X$. The CHF of $X$ can easily be obtained using the fact that the CHF of sum of independent random variables equals the product of the CHFs of the individual random variables $X_i$. The resulting CHF however in most cases does not adapt well to obtain the moments easily. To circumvent this difficulty, we relate the moments $M_l$ to the moments of the individual random variables $X_i$ in (1) which are denoted by $m_l$.

First, considering the sum of two independent random variables, i.e. $X = X_1 + X_2$, the moments of $X$ can be evaluated as follows:

$$M_n = E[X^n] = E[(X_1 + X_2)^n] = E\left[\sum_{k=0}^{n} \binom{n}{k} X_1^{\,k} X_2^{\,n-k}\right] = \sum_{k=0}^{n} \binom{n}{k} m_k^{(1)} m_{n-k}^{(2)} \tag{12}$$

where $m_k^{(1)}$ is the $k^{\text{th}}$ moment of $X_1$ and $m_{n-k}^{(2)}$ is the $(n-k)^{\text{th}}$ moment of $X_2$. Next, if $X$ is the sum of three random variables, i.e., $X = X_1 + X_2 + X_3$, we can first find the moments of the partial sum, $Y = (X_1 + X_2)$ applying (12) using MATLAB programme (Herniter, 2001) and then the moments of $X$ using the sum, $X = Y + X_3$ and applying (12) again as $X$ has been reduced to the sum of the two variables $Y$ and $X_3$. Consequently, considering sum of two random variables at a time, and using (12) repeatedly, the desired moments $M_n$ for the sum of finite number of independent random variables can be obtained very efficiently. We observe that even if the individual random variables are not identically distributed, the above procedure can be applied to determine the moments.

## CUMULATIVE DISTRIBUTION FUNCTION

The cumulative distribution function $F_X(x)$ is related to the PDF as

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \int_0^x f_X(u)\,du, & x \geq 0 \end{cases} \tag{13}$$

Substituting the modified Laguerre series in (8) into (13) and then replacing the Laguerre polynomial $L_k(x^2)$ by its series representation in (4), the resulting integral can be evaluated to obtain the CDF as

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \sum_{k=0}^{\infty} C_k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{1}{l!} \gamma(i+1, x^2) & x \geq 0 \end{cases} \tag{14}$$

where $\gamma(.)$ is the incomplete gamma function (Abramowitz and Stegum,1972; Andrews 1992 ).

## CONVERGENCE OF THE INFINITE SERIES

It is of interest to determine the convergence of the newly derived series. The convergence of the modified Laguerre series can be established using the following theorem.

**Theorem** (Lebedev,1965): If the real function $f_X(x)$ defined in the infinite interval $(0, \infty)$ is piecewise smooth in every finite subinterval $[x_1, x_2]$ where $0 < x_1 < x_2 < \infty$, and if the integral $\int_0^\infty e^{-x} x^\alpha f^2(x)\, dx$ is finite, then the series $f(x) = \sum_{n=0}^{\infty} C_n L_n^\alpha(x)$ , $0 < x < \infty$, with coefficients calculated from, $C_n = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^\infty e^{-x} x^\alpha f(x) L_n^\alpha(x)\, dx$, converges to $f_X(x)$ at every continuity point of $f_X(x)$. At a discontinuity point, the series convergence to $[f(x+0) + f(x-0)]/2$. Since all the PDFs under consideration are continuous and bounded, i.e. $0 < f_X(x) < K < \infty$ and $\int_0^\infty f_X(x) dx = 1$, it can easily be shown that $\int_0^\infty e^{-x} (f_X(x))^2\, dx < \infty$, thereby establishing the uniform convergence according to the above theorem.

## APPLICATION

As an example, we consider the evaluation of the PDF of the sum of Nakagami-$m$ distributions, which is a widely used probability model in communication engineering. The various signal fading conditions that frequently occur in wireless communication channels can accurately be modeled using the Nakagami distribution. Consequently, the sum of Nakagami random variables occurs in the performance analysis of fading communication systems. The Nakagami-$m$ PDF is given as (Proakis, 1995)

$$f_X(x_i) = \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega_i} \right)^m x_i^{2m-1} e^{-m x_i^2 / \Omega_i}, \quad x_i > 0, \tag{15}$$

where $\Omega$ is the second moment of $X$ and $m$ is the fading index. The parameter value $m$ determines the fading severity and, for example, the Nakagami density reduces to a Rayleigh density for $m=1$. The moments $M_n$ of the Nakagami-$m$ distribution can be evaluated to be

$$M_n = [2/\Gamma(m)](\Omega/m)^{n/2}\,\Gamma(m+n/2) \qquad (16)$$

The moments of the Nakagami sum and coefficients $C_k$ can be computed using the Nakagami moments in (16), with the method given Section IV. The PDFs and CDFs of the sum of 2 to 20 Nakagami-$m$ random variables are computed for two values of the fading index, $m=1$ (Rayleigh) and $m=2$, and depicted in Figures 1 to 4. These figures clearly demonstrate that the modified Laguerre series can be effectively used to determine the PDF of the sum of independent random variables for the sum of Rayleigh random variables, and also it is found that twenty five terms of the derived series are sufficient to yield accuracy up to five decimal places.

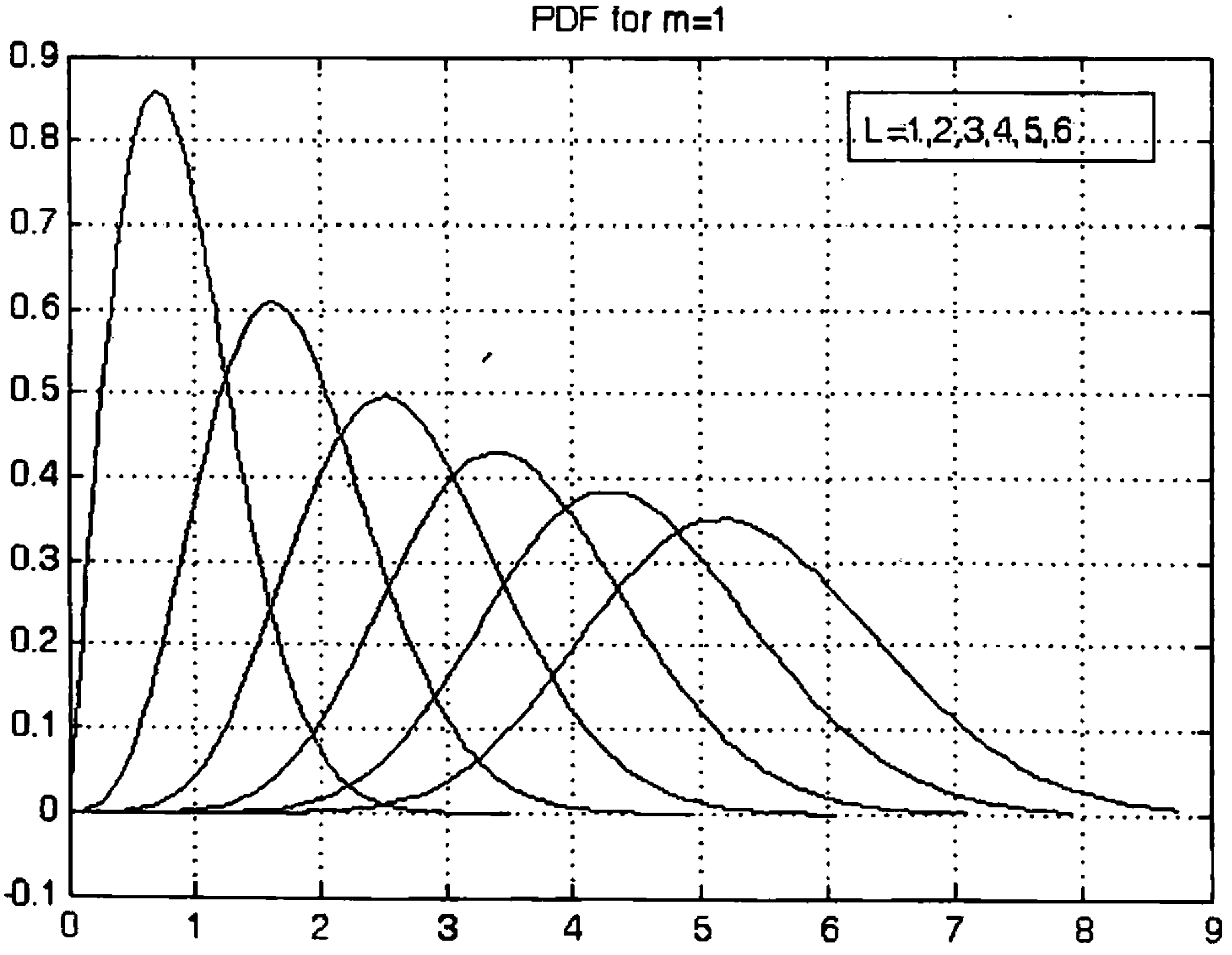

**Figure. 1:** The PDF of sum of $L$ Nakagami random variables (L=1,2,3,4,5,6; m=1, i.e. Rayleigh random variable)

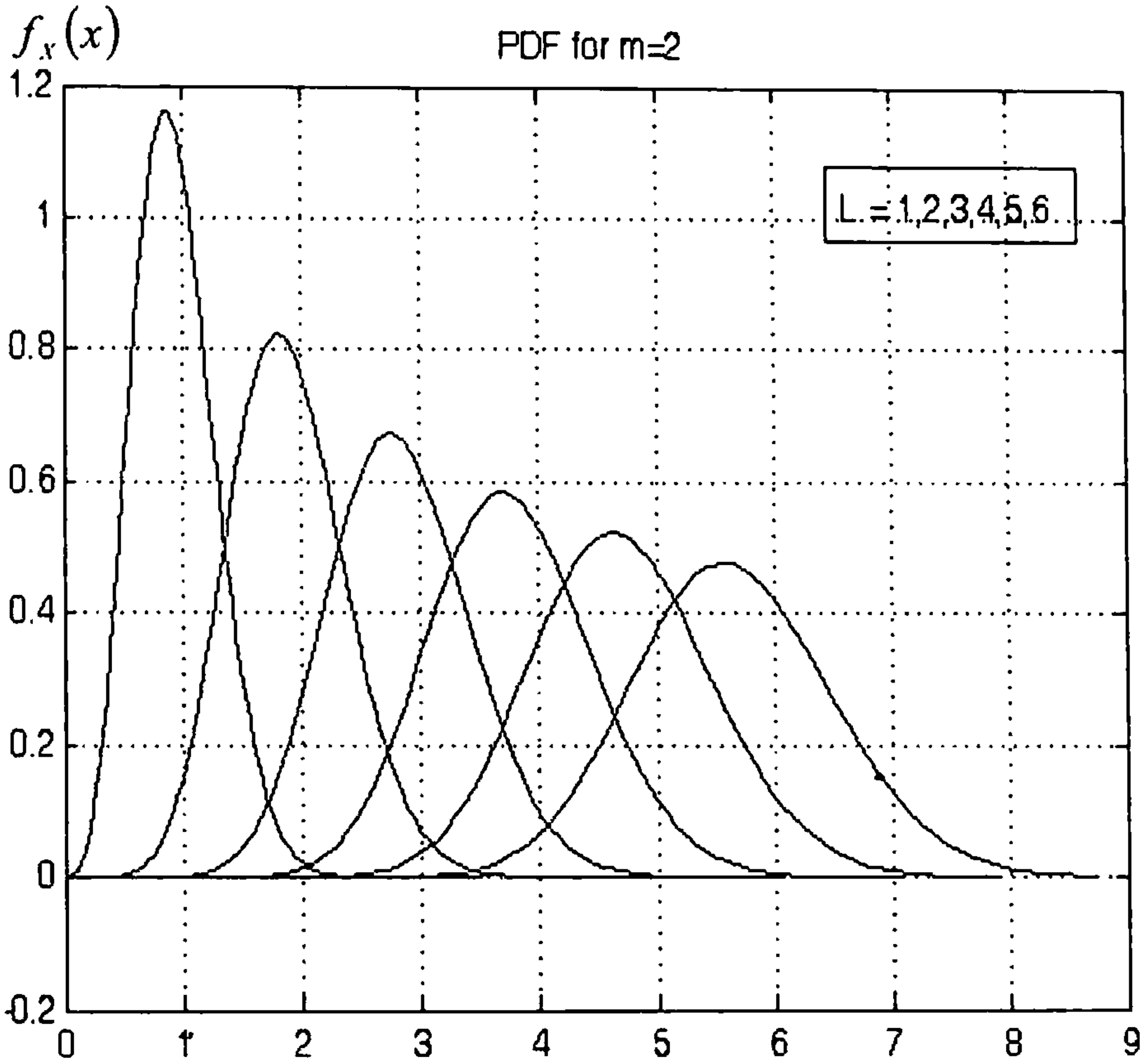

**Figure 2:** The PDF of sum of L Nakagami random variables (L=1, 2,3,4,5,6; m=2)

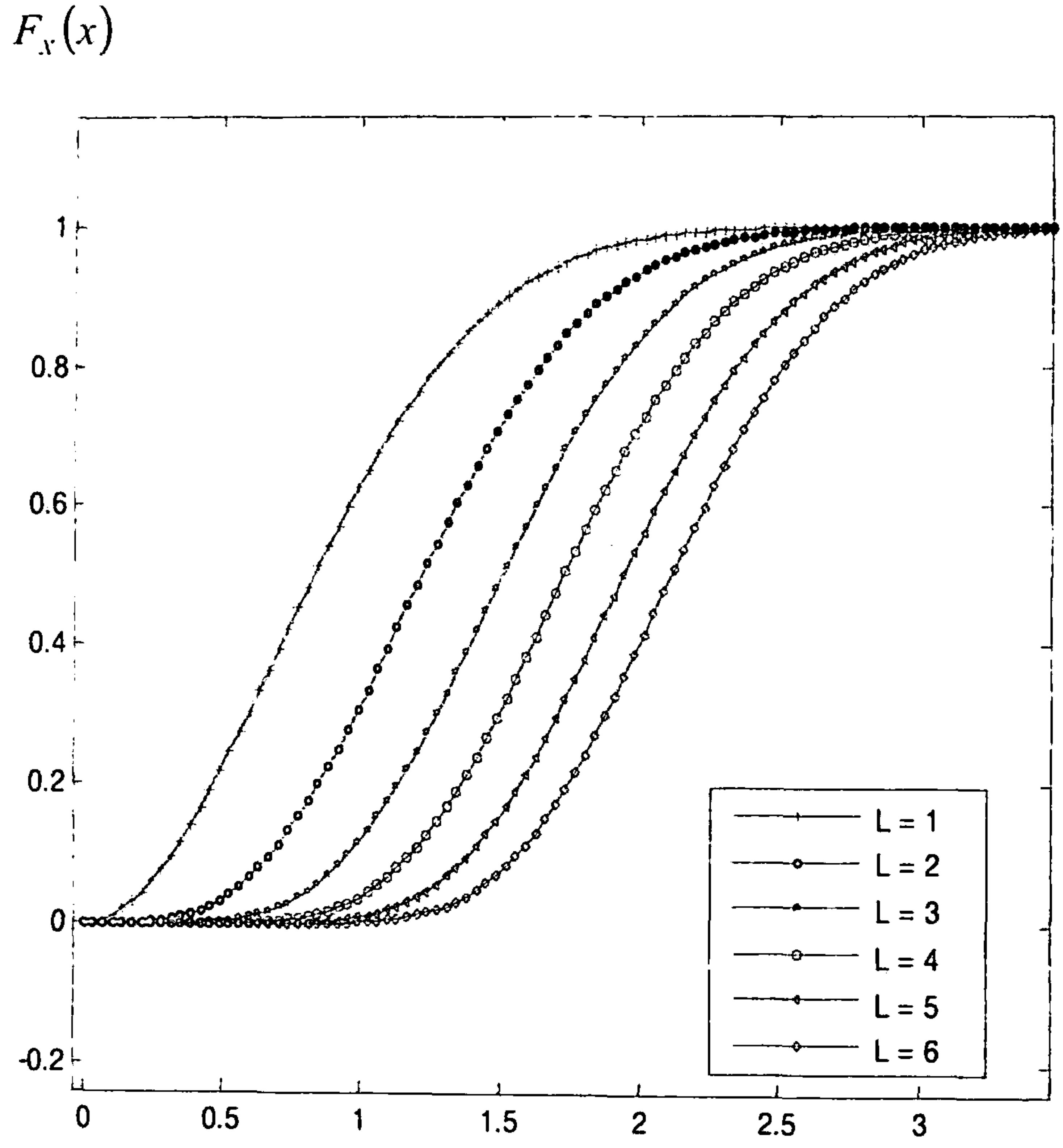

**Figure 3:** CDF, for the sum of Nakagami-m distributed rvs. M = 1, L is the number of rvs (i.e. Rayleigh random variable)

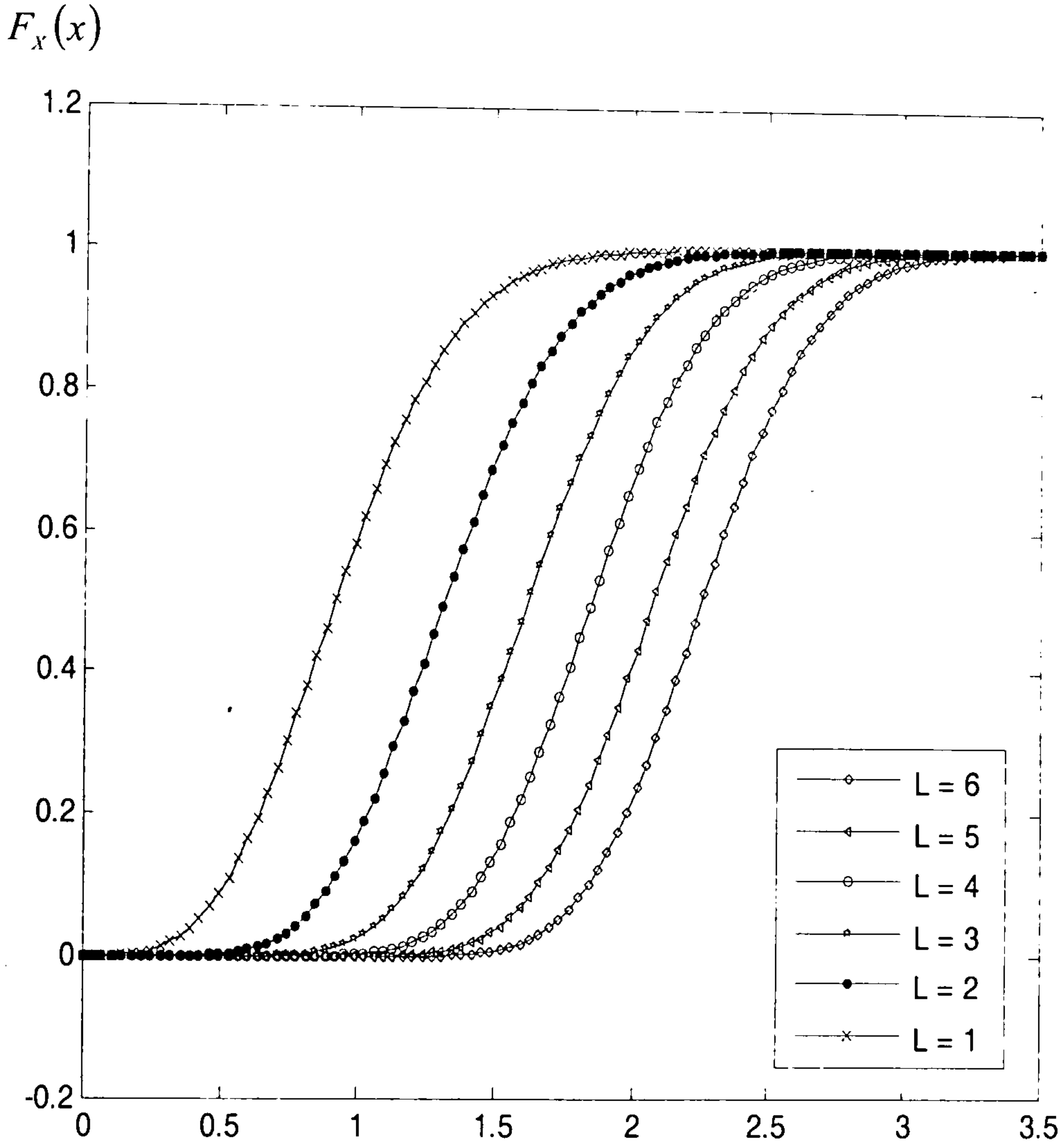

**Figure 4:** CDF, for the sum of Nakagami-m distributed rvs, m =2, L is the number of rvs

## CONCLUSION

We have derived an infinite series representation for the PDF of the sum of independent random variables using modified Laguerre polynomials. The weighting function has the form of a Rayleigh distribution, thus the series is exact for a Rayleigh distribution. The coefficients of the series have been expressed in terms of the moments of the constituent random variables. A method for efficient computation of the coefficients without using the characteristic function has been presented. The cumulative distribution function has also been expressed as an infinite series. To demonstrate the applicability of the newly derived series, the PDF and CDF of the sum of Nakagami random variables have been computed and graphically displayed.

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