APPENDIX II:

Uncertainty evaluation

A cylinder of nominal height 300 mm and diameter 100 mm was measured using a calibrated vernier caliper reading to 0.02 mm and a micrometer with the readability of 0.001 mm. The vernier was used to measure the height and the micrometer was used to measure the diameter. Each measurement was repeated 10 times

The calibration certificates for the caliper and micrometer state that the overall expanded uncertainty of measurement of the caliper and micrometer are 0.04 mm and 0.003mm respectively. The certificates further state that the expanded uncertainty has been arrived at multiplying the standard uncertainty by a factor of 2 (k=2) with 8 degrees of freedom.

The measurement model

$$V = \pi \cdot \left(\frac{\partial V}{\partial d}\right)^2 \cdot h$$

Where

d - diameter of the cylinder

h - height of the cylinder

Using equation 6.2.3 and assuming that the measurements of d and h are uncorrelated we get

$$U_c(V) = \sqrt{\left(\frac{\partial V}{\partial V}\right)^2 \cdot u^2(d) + \left(\frac{\partial V}{\partial h}\right)^2 \cdot u^2(h)}$$

(In this example, we have assumed π to be a constant with zero uncertainty.)

$$u_C(V) = \sqrt{\left(\frac{2\pi \cdot d \cdot h}{2}\right)^2 \cdot u^2(d) + \left(\frac{\pi \cdot d^2}{4}\right)^2 \cdot u^2(h)} = \sqrt{c_d^2 \cdot u^2(d) + c_h^2 \cdot u^2(h)}$$

where

$$c_d = \frac{2\pi \cdot d \cdot h}{2} \text{ and } c_h = \frac{\pi \cdot d^2}{4}$$

are the sensitivity coefficients of the measurements.

Test results

Test	d, mm	h, mm
1	100.001	300.00
2	100.001	300.00
3	100.000	300.00
4	100.000	300.02
5	99.999	300.00

Calculations

1. Diameter measurement

1.1 Mean diameter, $\bar{d}$ = 100.000 2 mm

1.2 Standard uncertainty due to repeatability, $u_{rep}(\bar{d}) = 0.000\ 374\ mm$ (equn:6.5.5)

1.3 Degrees of freedom, $\vartheta_{rep}(\bar{d}) = 5 - 1 = 4$

2. Height measurement

2.1 Mean height, $\bar{h} = 300.004\ mm$

2.2 Standard uncertainty due to repeatability, $u_{rep}(\bar{h}) = 0.004\ 00\ mm$ (equn:6.5.5)

2.3 Degrees of freedom, $\vartheta_{rep}(\bar{h}) = 5 - 1 = 4$

The volume of the cylinder is therefore given by

$$V = \pi \cdot \left(\frac{d}{2}\right)^2 \cdot h = \pi \cdot \left(\frac{100.0002}{2}\right)^2 \cdot 300.004\ \ mm^3 = 2356235\ \ mm^3$$

3. Uncertainty components

3.1 Standard uncertainty of diameter measurement, $u(d)$

The sources of uncertainty contributing to this measurement have been identified as:

- Calibration uncertainty of the micrometer
- Limited resolution of the micrometer
- Repeat measurement of diameter

Note: Other contributions such as effects due to temperature fluctuations, mis-alignment of the instrument with the work piece etc have been ignored. A proper evaluation would require that effects due to all possible source are identified and evaluated.

3.1.1. Standard uncertainty of calibration, $u_{cal}(d)$

$$u_{cal}(d) = \frac{U_{cal}(d)}{k} = \frac{0.003}{2}\,mm = 0.0015\ \ mm$$

with $\vartheta_{cal}(d) = 8$ degrees of freedom

3.1.2 Standard uncertainty due to resolution of the micrometer, $u_{res}(d)$

The micrometer has a resolution of 0.001 mm. Our readings would be estimated to the nearest 0.001 mm and assumed to be within an interval of 0.000 5 mm(square distribution). It is further assumed that this estimate is reliable to about 25% (ie. relative uncertainty of the assumed interval).

Therefore

$$u_{res}(d) = \frac{0.0005}{\sqrt{3}}\,mm = 0.000289\ \ mm$$

The degrees of freedom, $\vartheta_{res}(d)$ can be calculated using equation 6.5.11
Thus

$$\vartheta_{res}(d) = \frac{1}{2}\left(\frac{25}{100}\right)^{-2} = 8$$

Therefore, the standard uncertainty of diameter measurement is given by:

$$u(d) = \sqrt{u^2_{res}(\bar{d}) + u^2_{cal}(d) + u^2_{res}(d)} = \sqrt{0.000374^2 + 0.0015^2 + 0.000289^2}$$

$$= 0.00157 \quad mm$$

and the degrees of freedom is calculated using equation 6.5.12.

$$\vartheta(d) = \frac{u^4(d)}{\frac{u^4_{res}(\bar{d})}{\vartheta_{res}(\bar{d})} + \frac{u^4_{cal}(d)}{\vartheta_{cal}(d)} + \frac{u^4_{res}(d)}{\vartheta_{res}(d)}} = \frac{0.00157^4}{\frac{0.000374^4}{4} + \frac{0.0015^4}{8} + \frac{0.000289^4}{8}} = 9$$

Note: The degrees of freedom shall be rounded down to the nearest whole number.

The sensitivity coefficient c_d is given by

$$C_d = \frac{2\pi \cdot d \cdot h}{2} = \pi \cdot 100 \cdot 300 \quad mm^2 = 94247 mm^2$$

3.2 Standard uncertainty of height measurement, $u(h)$

3.2.1 Standard uncertainty of calibration, $u_{cal}(h)$

$$u_{cal}(h) = \frac{U_{cal}(h)}{k} = \frac{0.04}{2} mm = 0.02 \quad mm$$

$$\vartheta_{cal}(h) = 8$$

3.2.2 Standard uncertainty due to resolution of the caliper, $u_{res}(h)$

The caliper has a resolution of 0.02 mm. Our readings would be estimated to the nearest 0.02 mm and assumed to be within an interval of 0.01 mm. It is further assumed that this estimate is reliable to about 25% (ie. relative uncertainty of the assumed interval).

$$u_{res}(h) = \frac{0.01}{\sqrt{3}} mm = 0.00577 \quad mm$$

$$\vartheta_{res}(h) = \frac{1}{2}\left(\frac{25}{100}\right)^{-2} = 8$$

Therefore, the standard uncertainty of height measurement is given by:

$$u(h) = \sqrt{u_{rep}^2(h) + u_{cal}^2(h) + u_{res}^2(h)} = \sqrt{0.004472^2 + 0.002^2 + 0.00577^2} = 0.02129 \ mm$$

and the degrees of freedom is :

$$\vartheta(h) = \frac{u^4(h)}{\frac{u_{rep}^4(h)}{\vartheta_{rep}(h)} + \frac{u_{cal}^4(h)}{\vartheta_{cal}(h)} + \frac{u_{res}^4(h)}{\vartheta_{res}(h)}} = \frac{0.02129^4}{\frac{0.004472^2}{4} + \frac{0.02^4}{8} + \frac{0.00577^4}{8}} = 10$$

The sensitivity coefficient c_h is :

$$c_h = \frac{\pi \cdot d^2}{4} = \frac{\pi \cdot 100^2}{4} = 7853$$

Uncertainty budget

Quantity X_i	Estimate x_i	Estimated Standard uncertainty $u(x_i)$	Sensitivity coefficient c_i	Contribution to the standard uncertainty of the output, V $u_i(y) = c_i \cdot u(x_i)$
1.Diameter measurement		**0.001 57 mm**	**94 247 mm²**	**147 mm³**
1.1 Repeatability		0.000 374 *mm*		
1.2 Calibration	0.003 *mm*	0.001 5 *mm*		
1.3 Resolution	0.005 *mm*	0.000 289 *mm*		
2.Height measurement		**0.021 29 mm**	**7 853 mm²**	**167 mm³**
2.1 Repeatability		0.004 00 *mm*		
2.2 Calibration	0.04 *mm*	0.02 *mm*		
2.3 Resolution	0.01 *mm*	0.005 77 *mm*		

The combined standard uncertainty of volume measurement is therefore calculated as:

$$u_c(V) = \sqrt{147^2 + 167^2} = 223 \ mm^3$$

The effective degrees of freedom is therefore:

$$\vartheta_{eff}(V) = \frac{223^4}{\frac{147^4}{9} + \frac{167^4}{10}} = 19$$

The expanded uncertainty for a coverage factor *k*=2 *(95% confidence level)* is given by

$$U(V) = 2 \cdot 223 \ mm^3 = 446 \ mm^3$$

Reporting the result

There are many instances where after a careful measurement , the results are reported with so many figures which have no significance whatsoever in relation to the quantity measured. Therefore it is necessary to pay attention to individual measurements and the number of significant figures to which a value is reported in a series of measurements where the final value is obtained by using a mathematical formula.

In our example, we have measure two independent quantities viz*diameter* and the *height* of the cylinder. The diameter was measured using a micrometer reading to 0.001 mm whereas the height was measured using a caliper reading to 0.02 mm. It is evident from the set of results, that the diameter measurements have been reported to six(06) significant figures while the height measurements have five(05) significant figures retained in each measurement result. Therefore, the final result, the volume, should not have more than five(05) significant figures in its value.

The calculated value 2 356 235 mm^3 have 7 figures but only 5 are significant. Therefore we may report the volume as

$$V = 2356.2 \times 10^3 \quad mm^3$$

According to the accepted norms, the uncertainty is never reported to more than two significant figures. In our example, the expanded uncertainty U = 446 mm^3, which has three significant figures. Considering the fact that the value of volume has only one decimal place, we may report the uncertainty as U = 0.5 x 10^3 mm^3 (one significant figure).

The final expression of the value of the volume takes the form

$$V = (2356.2 \pm 0.5) \times 10^3 \quad mm^3$$