14 THE SCIENCE OF MEASUREMENTS

Rohan Perera
Division of Metrology, Sri Lanka Standards Institute, Colombo.

Measurement is a fundamental requirement in science. It provides the quantitative data, which can be statistically analysed, enabling the testing of hypothesis to draw conclusions. Today we practically measure everything we encounter: the weights of our consumer items, the volume of fuel, the distance we travel, temperature, pressure, humidity, light, current, voltage, power, speed, energy, etc. Reliable measurement is a cornerstone of research and the technologies, based on research findings. To ensure reliable measurements, a science has already developed to systematize and introduce standards in measurement. This is called Metrology.

What is Metrology ?

Metrology is the science of measurement. It includes units of measurement and their standards, measuring instruments and their field of application, and all theoretical and practical aspects relating to measurement, at any level of uncertainty, in any field of science and technology. Measurement science is not only for scientists alone. It is something that is of vital importance to all mankind. The intricate network of services, suppliers and communications upon which we are all dependent rely on metrology for their reliable and efficient operation. Inaccurate measurement results can lead to wrong decisions which can have serious consequences, costing money and even lives. The human and financial consequences of wrong decisions based on poor measurement results,when considered in matters as important as environmental change and pollution, can be catastrophic. It is important therefore to have reliable measurement results which are agreed and accepted by the stakeholders worldwide.

Metrology is broadly categorized in to three main fields: Scientific, Industrial and Legal metrology.

Scientific Metrology

This deals with aspects common to all metrological questions irrespective of the quantity measured. It covers general, theoretical and practical problems concerning units of measurement, including their realization and dissemination through scientific methods. It is also concerned with the problems of errors and uncertainties in measurement, and the problems of metrological properties of measuring instruments.

Industrial Metrology

This deals with measurements in production and quality control. It covers calibration procedures, calibration intervals, control of measurement processes and management of measuring instruments in industry to ensure that they are in a state of compliance with the requirements of their intended use.

Legal Metrology

This is concerned with legal and regulatory control of measurements. It is defined in the *International Vocabulary of Legal Metrology* as that part of metrology relating to activities which result from statutory requirements and concerns measurement, units of measurement, measuring instruments and methods of measurement, which are performed by competent bodies. It provides regulations for the control of measurements and measuring instruments in relation to protection of public safety, the environment, consumers and traders. It also plays an important role in fair trade.

History

Standardization of weights and measures has been a part of social and economic advancement since very early times. However, it was not until the 18th century that there was a unified system of measurement. The earliest system of weights and measures were based on human morphology. The names of units often referred to parts of human body: the hand, the foot, and the yard or cubit correspond to dimensions of human body parts. Consequently, these units of measurement were not fixed; they varied from one town to another, from one occupation to another and on the type of object to be measured.

This apparent lack of standardization was a source of error and fraud in social and commercial transactions. This acted as a barrier to international trade and also prevented the development of science as a global entity. With the rapid expansion of industry and trade, there was an increasing demand for the harmonization of weights and measures among countries. The politicians and scientists at the time resolved this situation by adopting a standard of measurement (distance or weight) by comparison with a standard taken from nature. The first such natural measure was the *meter*, which was defined as being equal to the *ten millionth part of one quarter of the terrestrial meridian*, but specified by measurements undertaken between Dunkerque and Barcelona. Such a unit was not arbitrary, being based on the size of the Earth.

The decimal metric system was introduced in France in 1795. The first standards of the meter and kilogram, against which all future copies were to be compared were deposited in the archives of the French republic in 1799 dedicated to *"all men and all times"*. However, the lack of uniformity in making copies and different countries maintaining their own standards in relation to the original copies were detrimental to international standardization. To overcome these difficulties, the *Bureau International des Poids et Measures* (BIPM) was founded by the terms of the diplomatic treaty known as the **Meter Convention** on 20th May 1875. Today this date is celebrated as the **World Metrology Day.**

Standards

The word *'standard'* is used to mean two different entities in measurement science. A widely adopted technical specification, recommendation or a similar document is called a Standard (product standard, standard specification etc). Similarly an entity used to define a unit of measurement is also called a Standard or to be precise, a Measurement Standard. A measurement standard can be a physical measure, measuring instrument, reference material or a measuring system intended to define, realize, conserve or reproduce a unit or one or more values of a quantity to serve as a reference.

Uncertainty of Measurement

General Principles

The objective of any measurement is to determine the value of the **measurand** that is the specific quantity subject to measurement. A measurement should therefore, begin with an appropriate specification of the measurand, the identification of the generic method of measurement in the form of a functional model, and a related measurement procedure.

No measurement is error free! The imperfections in the measurement system will give rise to errors in the result. Therefore, a measurement result is, at best only an approximation to the true value of the measurand and complete only if it is accompanied by a statement of uncertainty of the approximation.

Errors in the process of a measurement arise due to several sources. Some are random in nature and others follow a systematic pattern. The random effects give rise to possible random errors in the measurement process. The systematic effects give rise to possible systematic error in the measurement process.

Every time a measurement is repeated, under more or less the same conditions, random effects from various sources influence the measured value. A series of repeat measurements produces a scatter of values distributed around a ***mean value***.

A number of different sources contribute to variability, each time a measurement is taken, and their influence may be continually changing. They cannot be eliminated by the application of a correction, but the ***uncertainty*** in the ***mean value*** due to their effect may be reduced by increasing the number of repeat measurements.

An effect on a measurement result that may not have been included in the original specification of the measurand, but nevertheless influences the result may introduce a systematic error to the result. These remain unchanged when the measurement is repeated under near identical conditions. These types of errors can be corrected but since this correction is not exact, there will always be an associated uncertainty to the correction applied.

The two types of errors identified above have been grouped into ***Random Errors*** and ***Systematic Errors***. The uncertainty associated with each type of error is similarly grouped based on the method of evaluation. The evaluation of the uncertainties associated with random errors is classified as ***Type A Evaluation*** and evaluation of the uncertainties associated with systematic errors are classified as ***Type B Evaluation.*** This categorization based on the method of evaluation of uncertainty avoids certain ambiguities: a "random" component of uncertainty in one measurement may become a "systematic" component in another measurement that has its input result of the first measurement. For example, the overall uncertainty quoted on a calibration certificate for an instrument will include the components due to random effects but, when this overall value is subsequently used as the contribution in the evaluation of uncertainty in a test using that instrument, the contribution from calibration uncertainty would be regarded as systematic.

Type A Evaluation of uncertainty is by statistical calculation from a series of repeated measurements. The statistically estimated standard deviation of the measurements is then called ***Type A standard uncertainty***. Sometimes it is necessary to weigh the estimated standard deviation by a ***sensitivity coefficient***.

Type B Evaluation of uncertainty is by means other than that used for type A. For example, information about the sources of uncertainty may come from data in calibration certificates, from previous measurement data, from experience with the behavior of the instrument from the manufacturers' specifications and all other relevant information. A type B component is also characterized by estimated standard deviations and is called ***Type B standard uncertainty***. As stated before, sometimes it is necessary to weigh the estimated standard deviation by a ***sensitivity coefficient.***

All identical standard uncertainty components, irrespective of the method of evaluation, are combined to produce an overall value of uncertainty to be associated with the result of measurement. This is called the ***combined standard uncertainty***.

To meet the industrial, commercial, health and safety or other applications it is usually required to convert the combined standard uncertainty to an ***expanded uncertainty***, obtained by multiplying the combined standard uncertainty by a **coverage factor, k.** The expanded uncertainty provides a larger interval about the result of a measurement with a higher probability, that the value of the measurand lies within the stated interval.

Law of Propagation of Uncertainty

Generally a measurand Y is not a direct measurement but determined from N other quantities X_1, X_2,...., X_N. These other quantities have a functional relationship (model function) f with the output quantity Y and can be expressed in the form:

$$Y = f(X_1, X_2, \ldots, X_N) \qquad 14.1$$

The quantities X_i contain corrections or correction factors, as well as other quantities that take into account sources of variability such as different observers, instruments, sample laboratories and time at which the measurements are made. Therefore the function f should express not only a physical law but a *measurement process that includes all the quantities that contribute significantly to the overall uncertainty of measurement* of Y.

The estimate of the measurand or *output quantity* Y, denoted by 'y' is obtained from equation 14.1 *using input estimates* x_1, x_2,..., x_N for the values of N input quantities $X_1, X_2 \ldots, X_N$. Therefore the output estimate 'y', which is the result of the measurement, is given by:

$$y = f(x_1, x_2, \ldots, x_N) \qquad 14.2$$

The combined standard uncertainty of the measurement result 'y', denoted by $u_c(y)$ and taken to represent the *estimated standard deviation of the result*, is the positive square root of the estimated variance $u_c^2(y)$ given by the formula:

$$u_c^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \cdot u^2(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} \cdot u(x_i, x_j) \qquad 14.3$$

Equation 14.3 is based on a first-order Taylor series approximation of,

$$Y = f(X_1, X_2, \ldots.., X_N).$$

This is referred to as the *law of propagation of uncertainty*. The partial derivatives $\partial f/\partial x_i$ and $\partial f/\partial x_j$ are the sensitivity coefficients c_i and c_j. $u(x_i)$ is the standard uncertainty associated with the input estimate x_i and $u(x_i, x_j)$ is the ***estimated covariance*** associated with inputs x_i and x_j and can be expressed as:

$$u(x_i, x_j) = u(x_i) \cdot u(x_j) \cdot r(x_i \cdot x_j) \qquad (i \neq k) \qquad 14.3(a)$$

where, $r(x_i \cdot x_j)$ is referred to as the ***correlation coefficient*** of the input estimates x_i and x_j.

Then equation 14.3 can be expressed in the following form:

$$u_c^2(y) = \sum_{i=1}^{N} c_i^2 \cdot u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} c_i \cdot c_j \cdot u(x_i) \cdot u(x_j) \cdot r(x_i . x_j) \qquad 14.3\ (b)$$

OR

$$u_c^2(y) = \sum_{i=1}^{N} u_i^2(y) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} u_i(y) \cdot u_j(y) \cdot r(x_i . x_j) \qquad 14.3\ (c)$$

where $u_i(y) = c_i \cdot u(x_i)$; $u_j(y) = c_j \cdot u(x_j)$

If two input quantities X_i and X_j are correlated, that is, if they are mutually dependent in one way or another, their covariance has to be considered as a contribution to the uncertainty of the output Y. The ability to take into account the effect of correlations of input quantities depends on the knowledge of the measurement process and on the judgment of mutual dependency of input quantities. It is important to keep in mind that, in certain measurement processes, neglecting correlations between input quantities can lead to gross inaccuracies in the estimation of the standard uncertainty of the measurand.

The covariance associated with the estimates of two input quantities can be taken to be zero or insignificant, if:

- the input quantities are independent. For example, if the input quantities are repeatedly but not simultaneously observed in different independent measurements, or they represent resultant quantities of different evaluations that have been made independently

- either of the input quantities can be treated as constant
- investigation gives no information indicating the presence of correlation between the input quantities

Note: Correlation can sometimes be eliminated by the proper selection of the model function.

Uncertainty Budgets

The uncertainty analysis for a measurement, generally called the uncertainty budget of the measurement, should list all possible sources of uncertainty together with the associated estimated standard uncertainties and the methods (type -A or type - B) of evaluating them. For repeated measurements, the number of N observations also must be stated. It is always helpful to present the data in the form of a table with all quantities referenced by a symbol or a short identifier.

Table 1. An uncertainty budget

Quantity X_i	Estimate x_i	Estimated Standard uncertainty $u(x_i)$	Sensitivity coefficient c_i	Contribution to the standard uncertainty of the output $u_i(y) = c_i \cdot u(x_i)$
X_1	x_1	$u(x_1)$	c_1	$u_1(y)$
X_2	x_2	$u(x_2)$	c_2	$u_2(y)$
X_3	x_3	$u(x_3)$	c_3	$u_3(y)$
⋮	⋮	⋮	⋮	⋮
X_N	x_N	$u(x_N)$	c_N	$u_N(y)$
Y	y			$u_c(y)$

Evaluation of Uncertainty of Measurement of Input Estimates

Type A evaluation of standard uncertainty

Type A evaluation can be applied when a measurement is repeated under near identical conditions. If the measurement process has *sufficient resolution*, there will be an observable scatter of measurement results. The mean of several repeat measurements is the best estimate of the true mean of the population of the same measurement or the best estimate of the measurand. The mean of n repeat measurements is given by the formula:

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} \qquad 14.4$$

where x_i is the result of the i^{th} measurement.
The uncertainty of the estimate is evaluated by one of the following methods:

The experimental variance of the estimates x_i's is given by :

$$s^2(x) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \qquad 14.5$$

Its positive square root of $s^2(\bar{x})$ is the *experimental standard deviation* of the estimates. The best estimate of the variance of the mean of the measurements is given by the formula:

$$s^2(\bar{x}) = \frac{s^2(x)}{n} \qquad 14.6$$

The positive square root of $s^2(\bar{x})$ is called the *experimental standard deviation of the mean* and is designated as the standard uncertainty of the estimated mean $\bar{x}$, $u(\bar{x})$.

$$u(\bar{x}) = s(\bar{x}) \qquad 14.7$$

Combining equations 14.5, 14.6, and 14.7, gives the following expression for the estimated standard uncertainty of the mean of a set of repeated observations:

$$u(\bar{x}) = \sqrt{\frac{1}{n(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2} \qquad 14.8$$

Generally, when the number of measurements is low (n<10), the reliability of a Type A Evaluation of standard uncertainty as expressed by equation 14.8 needs to be considered. If the number of measurements cannot be increased, the following method could be used instead.

For a measurement that is well characterized and under statistical control, a combined or *pooled estimate of variance,* s_p^2 may be available that characterises the dispersion better than the estimated standard deviation obtained from a very limited number of observations as in equation 14.8.

In such a situation, the variance of the mean $s^2(\bar{x})$ may be estimated by:

$$s^2(\bar{x}) = \frac{s_p^2}{n} \qquad 14.9$$

The estimated standard uncertainty of the mean is then calculated as

$$u(\bar{x}) = \frac{s_p}{\sqrt{n}} \qquad 14.10$$

Type B Evaluation of Standard Uncertainty

The type B Evaluation of uncertainty, data may be derived from:

- previous measurement results,
- experience with or understanding of the behavior of properties of materials or instruments,
- manufacturer's specifications,
- data provided in calibration and other certificates, and
- uncertainties assigned to reference data taken from handbooks.

The use of available information in the Type B Evaluation of uncertainty calls for good understanding and experience in the measurement process. It is a skill that has to be learnt with experience. A properly assessed Type B Evaluation of standard uncertainty can be as good as, if not better than, a Type A Evaluation of standard uncertainty. This is clearly evident in the case where Type A Evaluation is based on a small number of observations.

In the Type B Evaluation, one must be aware of which one of the following conditions is most suitable for the measurement situation:

a) when only a single value is known for the quantity, a single measured value, a resultant value of a previous measurement, a reference value from literature, or a correction value, this value would be used for x. The standard uncertainty $u(x)$ associated with x is to be adopted where it is given. Otherwise it has to be evaluated from indisputable uncertainty data. If data of this kind is not available, the uncertainty has to be evaluated on the basis of experience.

b) When a probability distribution can be assumed for the quantity x based on theory or experience then the appropriate expected value and the square root of the variance of the distribution can be taken as the estimate and the standard uncertainty $u(x)$.

c) If only **upper** and **lower** limits (or containment limits) a_+ and a_- can be stimated for the quantity X (e.g. manufacturer's specification of the instrument, a temperature range, a rounding or truncation error resulting from automated data reduction), a probability distribution with constant probability density between these limits (rectangular probability distribution) has to be assumed for the possible variability of the input quantity X.

If we assume that the measurement situation warrants the assumption of a rectangular probability distribution for the variability of input data with definite upper and lower bounds a_+ and a_- then the best estimate for the value x_i may be calculated as:

$$x_i = \frac{1}{2} \cdot (a_+ + a_-) \qquad 14.11$$

The estimated variance $u^2(x_i)$ can be expressed as:

$$u^2(x_i) = \frac{1}{12} \cdot (a_+ - a_-)^2 \qquad 14.12$$

If we denote the range of the limits as $2a$ ('a' is called the semi-range), then

$$u^2(x_i) = \frac{1}{3} \cdot a^2 \qquad 14.13$$

The rectangular distribution is a reasonable description in terms of one's inadequate knowledge about the input quantity in the absence of any other information than the containment limits. But if it is known that the values of the quantity in question tend to be more towards the centre of the variability limits, a triangular or normal distribution may be a better model. On the other hand if values close to the variability limits are more likely than values near the centre, a U-shaped probability distribution would be more appropriate.

Table 2. Estimates and variance for a Type B evaluation

Assumed probability distribution	Range	Estimate x_i	Variance $u^2(x_i)$
Rectangular	$a_+ - a_-$	$\frac{1}{2} \cdot (a_+ + a_-)$	$\frac{1}{3} \cdot a^2$
Triangular	$a_+ - a_-$	$\frac{1}{2} \cdot (a_+ + a_-)$	$\frac{1}{6} \cdot a^2$
U-shape	$a_+ - a_-$	$\frac{1}{2} \cdot (a_+ + a_-)$	$\frac{1}{2} \cdot a^2$
Normal	$a_+ - a_-$	$\frac{1}{2} \cdot (a_+ + a_-)$	$\frac{1}{9} \cdot a^2$

Degrees of freedom

a) The degrees of freedom ϑ is equal to n - 1 for a single quantity estimated by the arithmetic mean of independent observations as described earlier.

b) For a Type B evaluation of uncertainty of the input estimate x_i, the degrees of freedom, ϑ_i associated with the estimated standard uncertainty $u(x_i)$ is given by the formula:

$$\vartheta_i = \frac{1}{2} \cdot \left[\frac{\Delta u(x_i)}{u(x_i)}\right]^{-2} \qquad 14.14$$

The quantity in large brackets is the relative uncertainty of $u(x_i)$.

Note: The quantity $\Delta u(x_i)$ may be considered as the uncertainty of the estimated uncertainty $u(x_i)$.

Example

If one's knowledge of how an input estimate was determined and how its standard uncertainty $u(x_i)$ was evaluated leads one to judge that the value of $u(x_i)$ is unreliable to about 25% of the estimated value $u(x_i)$

$$\frac{\Delta u(x_i)}{u(x_i)} = \frac{25}{100}$$

Then from equation 14.14,

$$\vartheta_i = \frac{1}{2} \cdot \left[\frac{25}{100}\right]^{-2} = 8$$

Similarly, if we assume a relative uncertainty of 50%, then

$$\vartheta_i = \frac{1}{2} \cdot \left[\frac{50}{100}\right]^{-2} = 2$$

c) The effective degrees of freedom ϑ_{eff} associated with the combined standard uncertainty of the output $u_c(y)$ is given by the formula

$$\vartheta_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^{n} \frac{u_i^4(y)}{\vartheta_i}} \qquad 14.15$$

with,

$$\vartheta_{eff} \leq \sum_{i=1}^{n} \vartheta_i$$

An example of the Uncertainty Evaluation of a cylinder is given in Appendix II.